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The globally stable solution of a stochastic nonlinear Schrödinger equation

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Abstract

Weak measurement of a subset of noncommuting observables of a quantum system can be modeled by the open-system evolution, governed by the master equation in the Lindblad form. The open-system density operator can be represented as a statistical mixture over non-unitarily evolving pure states, driven by the stochastic nonlinear Schrödinger equation (sNLSE). The globally stable solution of the sNLSE is obtained in the case where the measured subset of observables comprises the spectrum-generating algebra of the system. This solution is a generalized coherent state (GCS), associated with the algebra. The result is based on proving that the GCS minimizes the trace-norm of the covariance matrix, associated with the spectrum-generating algebra.

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1. Introduction

The number of solvable quantum dynamics models is quite limited. The importance of such models is that they form a source of insight into quantum phenomena. In addition, the solvable models are the starting point of approximate theories. An important class of solvable models is based on a Lie-algebraic Hamiltonian of the following form:

$$\hat{H} = \sum_j a_j \hat{X}_j \tag{1}$$

where the set $\{\hat{X}_j\}$ of observables is closed under the commutation relations:

$$[\hat{X}_i, \hat{X}_j] = \mathrm{i} \sum_{k=1}^{K} f_{ijk} \hat{X}_k, \tag{2}$$

i.e., it forms Lie algebra (Gilmore 1974) of the system operators, known as the spectrum-generating algebra (Bohm *et al* 1988) of the system.

The solvable nature of the dynamics generated by equation (1) motivates the search of transformation of complex many-body problems to an algebraic description. The next level of complexity is to add the binary term

$$\hat{H} = \sum_j a_j \hat{X}_j, + \sum_{kl} b_{kl} \hat{X}_k \hat{X}_l, \tag{3}$$

where $a_j = a_j^*$ and $b_{kl} = b_{lk}^*$. Such Lie-algebraic Hamiltonians are encountered in various fields of the many-body physics, such as molecular (Iachello and Levine 1995, Iachello 2006), nuclear (Bohm *et al* 1988, Iachello 2006) and condensed matter physics (Bohm *et al* 1988).

A useful property of the Lie-algebraic setting is a set of states which can be thought of as the most classical states with respect to measurement performed on the elements of the algebra. These states are termed generalized coherent states (GCS) with respect to the algebra (Perelomov 1985, Zhang *et al* 1990). An important property of the GCS is their invariance under the action of the unperturbed Hamiltonian, linear in the algebra elements, equation (1). This means that a GCS evolves into a GCS under the action of the Hamiltonian (1). In the perturbed case (3) GCS are generally no longer invariant but the GCS ansatz can be used for a mean-field approximation of the many-body dynamics (Kramer and Saraceno 1981, Zhang *et al* 1990).

Any realistic physical system is open, i.e., the interaction with the dissipating environment cannot be neglected. It is therefore of necessity to include the effect of an environment on the dynamics. Particularly interesting is the open-system evolution modeling the process of weak measurement (Diosi 2006). The open-system dynamics studied in the present work is generated by the Lindblad semi-group (Lindblad 1976, Breuer and Petruccione 2002):

$$\frac{\partial}{\partial t}\hat{\rho} = \mathcal{L}\hat{\rho} = -\mathrm{i}[\hat{H}, \hat{\rho}] - \sum_{j=1}^{K} \gamma_j [\hat{X}_j, [\hat{X}_j, \hat{\rho}]], \tag{4}$$

where the Hamiltonian has the algebraic form (1). The spectrum-generating algebra of the system spanned by $\{\hat{X}_i\}$ is assumed to be a compact semisimple algebra (Gilmore 1974) and the basis $\{\hat{X}_i\}$ is chosen to be orthonormal with respect to the Killing form (Gilmore 1974). The second term on the rhs of the Lindblad master equation (4) (the *dissipation term* in what follows) is responsible for the non-unitary character of the open-system evolution and is due to the system–bath interaction. This interaction can be interpreted as a process of weak measurement (Diosi 2006) of operators $\{\hat{X}_i\}_{i=1}^K$, performed on a quantum system, driven by the Hamiltonian (3), and the coupling constant γ_j reflects the strength of the measurement of the observable $\hat{X}_j$. It will be assumed that all the coupling constants are equal, $\gamma_j = \gamma$. This form of the dissipation term in equation (4) is chosen to ensure invariance under the group of unitary transformations, generated by spectrum-generating algebra, which is assumed to be the symmetry of the system–bath interaction. As a paradigm for such open-system dynamics (4) one may consider dynamics of a spin, immersed into an isotropic dissipating environment:

$$\frac{\partial}{\partial t}\hat{\rho} = -\mathrm{i}\omega[\hat{J}_1, \hat{\rho}] - \gamma \sum_{j=1}^{3} [\hat{J}_j, [\hat{J}_j, \hat{\rho}]], \tag{5}$$

where J_i satisfy the commutation relations of the $\mathfrak{su}(2)$: $[\hat{J}_i, \hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{J}_k$. Here, the $SU(2)$ group invariance of the dissipation term results from the isotropy of the system–bath interaction.

The solution of the master equation (4) can be represented as a statistical mixture of pure states, evolving according to the stochastic nonlinear Schrödinger equation (sNLSE) (Gisin 1984, Diosi 1988a, Gisin and Percival 1992):

$$d|\psi\rangle = \left\{ -\mathrm{i}\hat{H} - \sum_{i=1}^{K} \gamma_j (\hat{X}_i - \langle \hat{X}_i \rangle_\psi)^2 \right\} \mathrm{d}t|\psi\rangle + \sum_{i=1}^{K} (\hat{X}_i - \langle \hat{X}_i \rangle_\psi)\, \mathrm{d}\xi_i |\psi\rangle, \tag{6}$$

where the Wiener fluctuation terms $\mathrm{d}\xi_i$ satisfy

$$\langle \mathrm{d}\xi_i \rangle = 0, \qquad \mathrm{d}\xi_i\, \mathrm{d}\xi_j = 2\delta_{ij}\gamma_j\, \mathrm{d}t. \tag{7}$$

The purpose of the present work is to study the asymptotical properties of solutions of the sNLSE (6). We will show that the GCS associated with the spectrum-generating algebra of the system, driven by the Lindblad equation (4), are the globally stable solutions of the associated sNLSE (6). This property means that a group-invariant coupling to a dissipating environment will result in dynamics which can be represented as a statistical mixture of stable trajectories of the GCS, associated with the corresponding algebra.

2. Generalized coherent states and the total uncertainty

Let us assume that the algebra $\mathfrak{g}$ is represented irreducibly on the system's Hilbert space $\mathcal{H}$. Then an arbitrary state $\psi \in \mathcal{H}$ can be represented as a superposition of the *generalized coherent states (GCS)* (Perelomov 1985, Zhang *et al* 1990) $|\Omega, \psi_0\rangle$ with respect to the corresponding dynamical group G and an arbitrary state ψ_0:

$$|\psi\rangle = \int \mathrm{d}\mu(\Omega)|\Omega, \psi_0\rangle\langle\Omega, \psi_0|\psi\rangle, \tag{8}$$

where $\mu(\Omega)$ is the group-invariant measure on the coset space G/H (Gilmore 1974), $\Omega \in G/H$, $H \subset G$ is the maximal stability subgroup of the reference state ψ_0:

$$h|\psi_0\rangle = \mathrm{e}^{\mathrm{i}\phi(h)}|\psi_0\rangle, \qquad h \in H \tag{9}$$

and the GCS $|\Omega, \psi_0\rangle$ are defined as follows:

$$\hat{U}(g)|\psi_0\rangle = \hat{U}(\Omega h)|\psi_0\rangle = \mathrm{e}^{\mathrm{i}\phi(h)}\hat{U}(\Omega)|\psi_0\rangle$$
$$\equiv \mathrm{e}^{\mathrm{i}\phi(h)}|\Omega, \psi_0\rangle, \quad g \in G, \quad h \in H, \quad \Omega \in G/H, \tag{10}$$

where $\hat{U}(g)$ is a unitary transformation generated by a group element $g \in G$.

The group-invariant *total uncertainty* of a state with respect to a compact semisimple algebra $\mathfrak{g}$ is defined as (Delbourgo and Fox 1977, Perelomov 1985)

$$\Delta[\psi] \equiv \langle \hat{\Delta}_\psi \rangle_\psi = \sum_{j=1}^{K} \langle \hat{X}_j^2 \rangle_\psi - \sum_{j=1}^{K} \langle \hat{X}_j \rangle_\psi^2, \tag{11}$$

where we have used the notation $\hat{\Delta}_\psi \equiv \sum_i (\hat{X}_i - \langle \hat{X}_i \rangle_\psi)^2$. The first term on the rhs of equation (11) is the eigenvalue of the the Casimir operator of $\mathfrak{g}$ in the (irreducible) Hilbert space representation:

$$\hat{C} = \sum_{j=1}^{K} \hat{X}_j^2 \tag{12}$$

and the second term is the generalized purity (Barnum *et al* 2003) of the state with respect to $\mathfrak{g}$:

$$P_{\mathfrak{g}}[\psi] \equiv \sum_{j=1}^{K} \langle \hat{X}_j \rangle_\psi^2. \tag{13}$$

Let us define $\Delta_{\min}$ as a minimal total uncertainty of a quantum state and $c_{\mathcal{H}}$ as the eigenvalue of the Casimir operator of $\mathfrak{g}$ in the system Hilbert space. Then

$$\Delta_{\min} \leqslant \Delta[\psi] \leqslant c_{\mathcal{H}}, \tag{14}$$

The total uncertainty (11) is invariant under an arbitrary unitary transformation generated by $\mathfrak{g}$. Therefore, all the GCS with respect to the algebra $\mathfrak{g}$ and a reference state ψ_0 have a fixed value of the total invariance. It has been proved (Delbourgo and Fox 1977) that the minimal total uncertainty $\Delta_{\min}$ is obtained if and only if ψ_0 is the highest (or lowest) weight state of the representation. The value of $\Delta_{\min}$ is given by (Delbourgo and Fox 1977, Klyachko 2002)

$$\Delta_{\min} \equiv (\Lambda, \mu) \leqslant \Delta[\psi] \leqslant (\Lambda, \Lambda + \mu) = c_{\mathcal{H}}, \tag{15}$$

where $\Lambda \in \mathbb{R}^r$ is the highest weight of the representation, $\mu \in \mathbb{R}^r$ is the sum of the positive roots of $\mathfrak{g}$, r is the rank of $\mathfrak{g}$ (Gilmore 1974) and $(\ldots, \ldots)$ is the Euclidean scalar product in $\mathbb{R}^r$. The corresponding GCS were termed the generalized unentangled states with respect to the algebra $\mathfrak{g}$ (Barnum *et al* 2003, Klyachko 2002). The maximal value of the uncertainty is obtained in states termed maximally or completely entangled (Barnum *et al* 2003, Klyachko 2002) with respect to $\mathfrak{g}$. The maximum value equals $c_{\mathcal{H}}$ in the states having $\langle\psi|\hat{X}_j|\psi\rangle^2 = 0$ for all i. Such states exist in a generic irreducible representation (*irrep* in what follows) of an arbitrary compact simple algebra of observables (Klyachko 2002). Generic superpositions of the GCS have larger uncertainty and are termed generalized entangled states with respect to $\mathfrak{g}$ (Barnum *et al* 2003, Klyachko 2002). In what follows, it is assumed that the reference state ψ_0 for the GCS minimizes the total invariance (11).

3. The main result: global stability of the generalized coherent states

3.1. The time evolution of the total uncertainty

The time evolution of the total uncertainty (11) of a pure state evolving according to the sNLSE (6) can be calculated as follows:

$$\begin{aligned} \mathrm{d}\Delta[\psi(t)] &= \mathrm{d}\sum_i \left(\langle\hat{X}_i^2\rangle_\psi - \langle\hat{X}_i\rangle_\psi^2\right) = -\mathrm{d}\sum_i \langle\hat{X}_i\rangle_\psi^2 \\ &= -\sum_i (2\,\mathrm{d}\langle\hat{X}_i\rangle_\psi\langle\hat{X}_i\rangle_\psi + \mathrm{d}\langle\hat{X}_i\rangle_\psi\,\mathrm{d}\langle\hat{X}_i\rangle_\psi), \end{aligned} \tag{16}$$

where we have used prescription of the Ito calculus: $\mathrm{d}(xy) = \mathrm{d}x\,y + x\,\mathrm{d}y + \mathrm{d}x\,\mathrm{d}y$ and the fact that $\mathrm{d}\sum_i \langle\hat{X}_i^2\rangle_\psi = 0$ by the invariance of the Casimir operator (12) under dynamics in an irreducible representation. To calculate $\mathrm{d}\hat{X}_i$ we derive the Heisenberg equations of motion, corresponding to the sNLSE (6).

Equation (6) is equivalent to the following equation for the corresponding projector $\hat{P}_\psi = |\psi\rangle\langle\psi|$:

$$\mathrm{d}\hat{P}_\psi = \left(-\mathrm{i}[\hat{H}, \hat{P}_\psi] - \gamma\sum_{j=1}^K [\hat{X}_j, [\hat{X}_j, \hat{P}_\psi]]\right)\mathrm{d}t + \sum_i \{(\hat{X}_i - \langle\hat{X}_i\rangle_\psi)\,\mathrm{d}\xi_i, \hat{P}_\psi\}. \tag{17}$$

Equation (17) implies that the following stochastic Heisenberg equation can be used to calculate the increment $\mathrm{d}\langle\hat{X}_i\rangle$ for an arbitrary operator $\hat{X}_i$:

$$\begin{aligned} \mathrm{d}\hat{X}_i &= \left(\mathrm{i}[\hat{H}, \hat{X}_i] - \gamma\sum_{j=1}^K [\hat{X}_j, [\hat{X}_j, \hat{X}_i]]\right)\mathrm{d}t + \sum_j \{(\hat{X}_j - \langle\hat{X}_j\rangle_\psi)\,\mathrm{d}\xi_j, \hat{X}_i\} \\ &= (\mathrm{i}[\hat{H}, \hat{X}_i] - \gamma c_{\mathtt{adj}}\hat{X}_i)\,\mathrm{d}t + \sum_j \{(\hat{X}_j - \langle\hat{X}_j\rangle_\psi)\,\mathrm{d}\xi_j, \hat{X}_i\}, \end{aligned} \tag{18}$$

where $c_{\mathtt{adj}}$ is the quadratic Casimir in the adjoint representation (see equation (12)). Multiplying equation (18) by $\langle \hat{X}_i \rangle_\psi$, summing up over all the observables and computing the expectation value we obtain

$$\begin{aligned}
\sum_{i=1}^{K} \langle \hat{X}_i \rangle_\psi \, \mathrm{d}\langle \hat{X}_i \rangle &= \left(\mathrm{i} \sum_{i=1}^{K} \langle \hat{X}_i \rangle_\psi \langle [\hat{H}, \hat{X}_i] \rangle_\psi - \gamma c_{\mathtt{adj}} \sum_{i=1}^{K} \langle \hat{X}_i \rangle_\psi^2 \right) \mathrm{d}t \\
&\quad + \sum_{j,i} \langle \hat{X}_i \rangle_\psi \langle \{ (\hat{X}_j - \langle \hat{X}_j \rangle_\psi) \, \mathrm{d}\xi_j, \hat{X}_i \} \rangle_\psi \\
&= -\gamma c_{\mathtt{adj}} \sum_{i=1}^{K} \langle \hat{X}_i \rangle_\psi^2 \, \mathrm{d}t \\
&\quad + \sum_{i,j=1}^{K} \langle \hat{X}_i \rangle_\psi (\langle \{ \hat{X}_j, \hat{X}_i \} \rangle_\psi - 2 \langle \hat{X}_j \rangle_\psi \langle \hat{X}_i \rangle_\psi) \, \mathrm{d}\xi_j,
\end{aligned} \tag{19}$$

where the contribution of the Hamiltonian term has vanished due to the antisymmetry of the structure constants f_{jik} of $\mathfrak{g}$:

$$\begin{aligned}
\mathrm{i} \sum_{i,j=1}^{K} a_j \langle \hat{X}_i \rangle_\psi \langle [\hat{X}_j, \hat{X}_i] \rangle_\psi &= \mathrm{i} \sum_{i,j,k=1}^{K} a_j \langle \hat{X}_i \rangle_\psi \langle \mathrm{i} f_{jik} \hat{X}_k \rangle_\psi \\
&= -\sum_{j=1}^{K} a_j \sum_{i,k=1}^{K} f_{jik} \langle \hat{X}_i \rangle_\psi \langle \hat{X}_k \rangle_\psi = 0.
\end{aligned} \tag{20}$$

From equation (18) we get

$$\begin{aligned}
\mathrm{d}\langle \hat{X}_i \rangle_\psi d \langle \hat{X}_i \rangle_\psi &= \sum_{k,l} \mathrm{d}\xi_k \, \mathrm{d}\xi_l \langle \{ (\hat{X}_k - \langle \hat{X}_k \rangle_\psi), \hat{X}_i \} \rangle_\psi \langle \{ (\hat{X}_l - \langle \hat{X}_l \rangle_\psi), \hat{X}_i \} \rangle_\psi \\
&= 2\gamma \, \mathrm{d}t \sum_k \langle \{ (\hat{X}_k - \langle \hat{X}_k \rangle_\psi), \hat{X}_i \} \rangle_\psi^2 \\
&= 2\gamma \, \mathrm{d}t \sum_k (\langle \{ \hat{X}_k, \hat{X}_i \} \rangle_\psi - 2 \langle \hat{X}_k \rangle_\psi \langle \hat{X}_i \rangle_\psi)^2.
\end{aligned} \tag{21}$$

Inserting equations (21) and (20) into equation (16) we obtain

$$\begin{aligned}
\mathrm{d}\langle \hat{\Delta} \rangle_\psi &= -\sum_i (2 \, \mathrm{d}\langle \hat{X}_i \rangle_\psi \langle \hat{X}_i \rangle_\psi + \mathrm{d}\langle \hat{X}_i \rangle_\psi \, \mathrm{d}\langle \hat{X}_i \rangle_\psi) \\
&= 2\gamma \left(c_{\mathtt{adj}} \sum_{i=1}^{K} \langle \hat{X}_i \rangle_\psi^2 - \sum_{k,i} (\langle \{ \hat{X}_k, \hat{X}_i \} \rangle_\psi - 2 \langle \hat{X}_k \rangle_\psi \langle \hat{X}_i \rangle_\psi)^2 \right) \mathrm{d}t \\
&\quad - 2 \sum_{j,i} \langle \hat{X}_i \rangle_\psi (\langle \{ \hat{X}_j, \hat{X}_i \} \rangle_\psi - 2 \langle \hat{X}_j \rangle_\psi \langle \hat{X}_i \rangle_\psi) \, \mathrm{d}\xi_j.
\end{aligned} \tag{22}$$

The remaining terms in equation (22) describe the effect of the bath (weak measurement) on the total uncertainty of a pure state evolving according to the sNLSE. It can be shown by direct calculation that these terms vanish in a GCS. But a simpler way to show this is to note that the infinitesimal evolution of the state, corresponding to the sNLSE (6) dropping the

Hamiltonian term, is given by

$$|\psi\rangle + |d\psi\rangle = \exp\left\{-2\gamma\hat{\Delta}_\psi\,\mathrm{d}t + \sum_i (\hat{X}_i - \langle\hat{X}_i\rangle_\psi)\,\mathrm{d}\xi_i\right\}|\psi\rangle$$
$$= \exp\left\{\sum_i (\hat{X}_i - \langle\hat{X}_i\rangle_\psi)\,\mathrm{d}\xi_i\right\}\exp\{-2\gamma\hat{\Delta}_\psi\,\mathrm{d}t\}|\psi\rangle$$
$$= \exp\{\phi(t)\}\exp\left\{\sum_i (\hat{X}_i - \langle\hat{X}_i\rangle_\psi)\,\mathrm{d}\xi_i\right\}|\psi\rangle, \qquad (23)$$

where we have used the fact (Delbourgo and Fox 1977) that a GCS is an eigenstate of the operator $\hat{\Delta}_\psi$, defined after equation (11). From equation (23) we see that the infinitesimal transformation of the state is driven by the operator *linear* in the generators of the algebra. Therefore, a GCS transforms into a GCS under the infinitesimal evolution[1] and the total uncertainty of the evolving state remains constant (and minimal).

The first term in equation (22), considered as a functional on the Hilbert space, has global maximum in the GCS (Subsection C). Therefore, on average, the rate of uncertainty loss (termed *localization* rate) is minimal in a GCS. In a GCS the second (stochastic) term vanishes. Since the rate of localization is zero in a GCS, as proved above, it follows that the average rate of localization obtains minimum at zero. Therefore, an arbitrary state localizes on average. Since the uncertainty is minimal in a GCS, the localization on average implies that almost every solution of the sNLSE (6) approaches asymptotically a GCS.

3.2. An illustration. $\mathfrak{su}(2)$*-case*

For the purpose of illustration let us consider a quantum system, driven by a $\mathfrak{su}(2)$-algebraic Hamiltonian $\hat{H} = \omega\hat{J}_1$, i.e., a spin. The GCS, associated with the $\mathfrak{su}(2)$ algebra, are the so-called spin-coherent states (Arecchi *et al* 1972) which are characterized by the maximal projection of the spin. A spin-coherent state of the system in the irreducible $(2j+1)$-dimensional Hilbert space representation has the group-invariant uncertainty (11) equal to j, the spin quantum number of the representation. A superposition of GCS has larger uncertainty. For example, a superposition $(|-j\rangle + |j\rangle)/\sqrt{2}$ has uncertainty equal to the maximal possible value $j(j+1)$, i.e., the eigenvalue of the Casimir operator. This follows from the fact that the projection of the spin in this state in any direction vanishes and, as a consequence, its generalized purity (13) is zero.

The Lindblad master equation, corresponding to the weak measurement of the projections of the spin $\hat{J}_i$, is equation (5). Taking $(|-j\rangle+|j\rangle)/\sqrt{2}$ for the initial state, we expect to observe stochastic evolution of a pure-state unraveling of equation (5) according to the corresponding sNLSE (6), approaching asymptotically a spin-coherent state. The spin coherent state will have minimal uncertainty and maximal amplitude of the spin projection.

Figure 1 shows the evolution of the normalized expectation values of the spin projections $\langle\hat{J}_i\rangle/j$, $i = 1, 2, 3$, for the pure state $\psi(t)$ evolving according to the sNLSE (6), associated with the master equation (5). Initial state of the system is $(|-j\rangle + |j\rangle)/\sqrt{2}$ and the total spin quantum number is $j = 16$. In addition, the normalized total uncertainty $\Delta[\psi]/j^2$ is

[1] This fact does not follow directly from the definition of the GCS, since the evolution in equation (23) is not unitary. Nonetheless, by an application of the Baker–Campbell–Hausdorff disentangling formula (Zhang *et al* 1990) it can be shown that $\forall\,\zeta_i \in \mathbb{C}$ and $\hat{X}_i \in \mathfrak{g}\exp(\sum_i \zeta_i\hat{X}_i)|\Omega,\Lambda\rangle \propto \exp(\sum_\alpha \eta_\alpha\hat{E}_{-\alpha})|\Omega,\Lambda\rangle$ ($\hat{E}_{-\alpha}$ is the lowering operator of the algebra, corresponding to the positive root α), which is a GCS up to a normalization [see Kramer and Saraceno (1981), pp 31–2 for details].

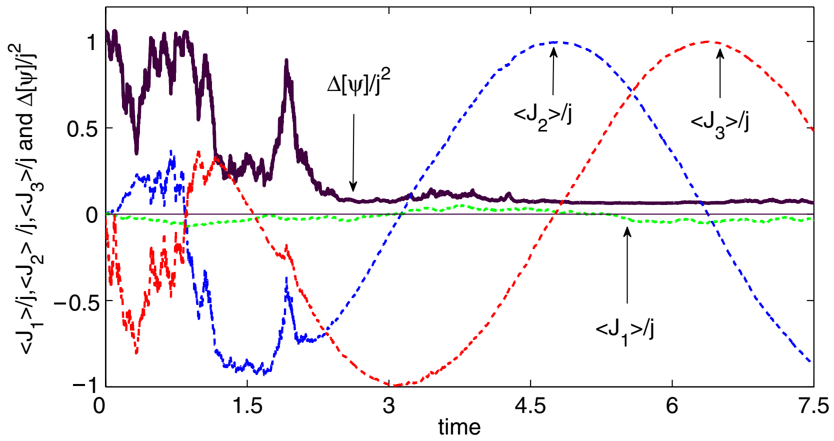

Figure 1. The normalized expectation values of the spin projections $\langle\hat{J}_i\rangle/j$, $i = 1, 2, 3$ (dashed lines), and the normalized total uncertainty $\Delta[\psi]/j^2$ (solid line) as a function of time. The pure state $\psi(t)$ evolves according to the sNLSE (6), corresponding to the master equation (5). Time is measured in units of ω^{-1} and the spin–bath coupling is $\gamma = \omega/160$. Initial state of the system is $(|-j\rangle + |j\rangle)/\sqrt{2}$ and the total spin quantum number is $j = 16$. The asymptotic solution has minimal (normalized) uncertainty of $1/16$, i.e., it is a spin-coherent state.

plotted as a function of time. The stochastic evolution asymptotically leads to a state having constant value of the (normalized) uncertainty equal to $1/16$, i.e., the minimal value in the representation. It follows therefore that the asymptotic solution is a spin-coherent state.

3.3. The proof of the main result

Next we prove that the first term in equation (22), considered as a functional on the Hilbert space, has global maximum in the GCS. The first sum in this term is just the generalized purity of the state (13), which has a global maximum in a GCS (Barnum *et al* 2003, Klyachko 2008), while the second sum is the trace-norm of the covariance matrix, which obtains global minimum in a GCS.

Theorem. *The trace-norm of the covariance matrix* $M_{ij} = \langle\{\hat{X}_i, \hat{X}_j\}\rangle_\psi - 2\langle\hat{X}_i\rangle_\psi\langle\hat{X}_j\rangle_\psi$ *is minimal in a maximal (minimal) weight state of the irrep, i.e., in a GCS.*

Proof. The trace-norm is invariant under unitary transformations, generated by the algebra $\mathfrak{g}$. Therefore, any orthonormal basis $\hat{X}_i$ can be used for the calculation of the trace-norm. Consider a particular choice of the basis $\hat{X}_i$ such that the projection of the density operator $\hat{\rho} = |\psi\rangle\langle\psi|$ on $\mathfrak{g}$ is contained in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let us use index i, j for the elements of $\mathfrak{h}$ and α, β for the elements of the root subspace. Then,

$$\mathtt{Tr}\{M^2\} = \sum_{i,j} M_{i,j}^2 + \sum_{i,\alpha} M_{i,\alpha}^2 + \sum_{i,\alpha} M_{\alpha,i}^2 + \sum_{|\alpha|\neq|\beta|} M_{\alpha,\beta}^2 + \sum_{|\alpha|=|\beta|} M_{\alpha,\beta}^2. \qquad (24)$$

Let us focus on the last term in equation (24). Since the projection of the state on $\mathfrak{g}$ is contained in the Cartan subalgebra, it vanishes on the root subspace, i.e., $\langle\hat{X}_\alpha\rangle = 0$, for every α. Then

$$\sum_{|\alpha|=|\beta|} M_{\alpha,\beta}^2 = \sum_{|\alpha|=|\beta|} \langle\{\hat{X}_\alpha, \hat{X}_\beta\}\rangle_\psi^2. \qquad (25)$$

Using notation $E_{\pm\alpha}$ for the raising and the lowering operators of the algebra, corresponding to the positive root α we obtain

$$\begin{aligned}\sum_{|\alpha|=|\beta|} M^2_{\alpha,\beta} = \sum_{|\alpha|=|\beta|} \langle\{\hat{X}_\alpha, \hat{X}_\beta\}\rangle^2_\psi &= -\frac{1}{2}\sum_{\alpha>0}\langle(\hat{E}_\alpha + \hat{E}_{-\alpha})(\hat{E}_\alpha - \hat{E}_{-\alpha}) \\ &\quad + (\hat{E}_\alpha - \hat{E}_{-\alpha})(\hat{E}_\alpha + \hat{E}_{-\alpha})\rangle^2 + \sum_{\alpha>0}\langle(\hat{E}_\alpha + \hat{E}_{-\alpha})^2\rangle^2 + \sum_{\alpha>0}\langle(\hat{E}_\alpha - \hat{E}_{-\alpha})^2\rangle^2 \\ &= -2\sum_{\alpha>0}\left\langle\hat{E}^2_\alpha - \hat{E}^2_{-\alpha}\right\rangle^2 + \sum_{\alpha>0}\left\langle\hat{E}^2_\alpha + \hat{E}^2_{-\alpha} + \hat{E}_\alpha\hat{E}_{-\alpha} + \hat{E}_{-\alpha}\hat{E}_\alpha\right\rangle^2 \\ &\quad + \sum_{\alpha>0}\left\langle\hat{E}^2_\alpha + \hat{E}^2_{-\alpha} - \hat{E}_\alpha\hat{E}_{-\alpha} - \hat{E}_{-\alpha}\hat{E}_\alpha\right\rangle^2 \\ &= -2\sum_{\alpha>0}\left\langle\hat{E}^2_\alpha - \hat{E}^2_{-\alpha}\right\rangle^2 + 2\sum_{\alpha>0}\left\langle\hat{E}^2_\alpha + \hat{E}^2_{-\alpha}\right\rangle^2 + 2\sum_{\alpha>0}\langle\hat{E}_\alpha\hat{E}_{-\alpha} + \hat{E}_{-\alpha}\hat{E}_\alpha\rangle^2 \\ &= 8\sum_{\alpha>0}\left\langle\hat{E}^2_\alpha\right\rangle\langle\hat{E}^2_{-\alpha}\rangle + 2\sum_{\alpha>0}\langle\hat{E}_\alpha\hat{E}_{-\alpha} + \hat{E}_{-\alpha}\hat{E}_\alpha\rangle^2. \end{aligned} \tag{26}$$

The density operator $\hat{\rho} = |\psi\rangle\langle\psi|$ can be expressed in the basis of the eigenstates $|\mu\rangle$ of the Cartan operators $\hat{X}_i \in \mathfrak{h}$, $\hat{X}_i|\mu\rangle = \mu_i|\mu\rangle$

$$\hat{\rho} = \sum_{\mu,\mu'} c_\mu c^*_{\mu'}|\mu\rangle\langle\mu'|. \tag{27}$$

Then the last term in equation (26) obtains

$$\begin{aligned}2\sum_{\alpha>0}\langle\hat{E}_\alpha\hat{E}_{-\alpha} + \hat{E}_{-\alpha}\hat{E}_\alpha\rangle^2 &= 2\sum_{\alpha>0}\left(\sum_{\mu,\mu'} c_\mu c^*_{\mu'}\langle\mu'|\hat{E}_\alpha\hat{E}_{-\alpha} + \hat{E}_{-\alpha}\hat{E}_\alpha|\mu\rangle\right)^2 \\ &= 2\sum_{\alpha>0}\left(\sum_\mu |c_\mu|^2\langle\mu|\hat{E}_\alpha\hat{E}_{-\alpha} + \hat{E}_{-\alpha}\hat{E}_\alpha|\mu\rangle\right)^2. \end{aligned} \tag{28}$$

States $|\mu + k\alpha\rangle$ form an irreducible representation of the $\mathfrak{su}(2)$, spanned by

$$E^\pm \equiv E_{\pm\alpha}/|\alpha| \qquad E_3 \equiv \alpha\cdot\hat{H}/|\alpha|^2, \qquad \hat{H}_i \equiv \hat{X}_i \in \mathfrak{h} \tag{29}$$

obeying $\mathfrak{su}(2)$ commutation relations (Georgi 1982)

$$[E_3, E^\pm] = \pm E^\pm; \qquad [E^+, E^-] = E_3. \tag{30}$$

Therefore, the state $|\mu\rangle$ can be labeled as $|m_\alpha, j_\alpha\rangle$, where j_α is the maximal weight of the corresponding irrep of the $\mathfrak{su}(2)$ and m_α is the weight, corresponding to the state $|\mu\rangle$ in the irrep. Then,

$$\begin{aligned}\langle\hat{E}_{-\alpha}\hat{E}_\alpha + \hat{E}_\alpha\hat{E}_{-\alpha}\rangle^2_\psi = |\alpha|^4\langle 2\hat{E}^-\hat{E}^+ + \hat{E}_3\rangle^2_\psi &= |\alpha|^4\langle m_\alpha, j_\alpha|2\hat{E}^-\hat{E}^+ + \hat{E}_3|m_\alpha, j_\alpha\rangle^2 \\ &= |\alpha|^4(j_\alpha + j^2_\alpha - m^2_\alpha)^2. \end{aligned} \tag{31}$$

The term (31) obtains minimum in the maximal (minimal) weight state of the j_α irrep, corresponding to $m_\alpha = j_\alpha(m_\alpha = -j_\alpha)$. Therefore, $|m_\alpha, j_\alpha\rangle$ is annihilated by the $E^+(E^-)$, and, by equations (29), the state $|\mu\rangle$ is annihilated by $E_\alpha(E_{-\alpha})$. The minimum of the sum (29) is obtained in the state, annihilated by $E_\alpha(E_{-\alpha})$ for all positive roots α, i.e., in the maximal (minimal) weight state $\hat{\rho} = |\Lambda\rangle\langle\Lambda|$. The first term in equation (26) is nonnegative and vanishes at $|\psi\rangle = |\Lambda\rangle$, therefore it obtains minimum at $|\Lambda\rangle$. Therefore, the term (26) in the sum (24) obtains minimum at $|\Lambda\rangle$. Since $|\Lambda\rangle$ is an eigenstate of every Cartan operator $\hat{X}_i$, the first term in equation (24) vanishes at $|\Lambda\rangle$. For the same reason and the fact that the projection of $\hat{\rho}$ on the root subspace vanishes both the second and the third terms in equation (24) also vanish at $|\Lambda\rangle$. The fourth term in equation (24) vanishes at $|\Lambda\rangle$ since $\langle\Lambda|\hat{E}_\alpha\hat{E}_\beta|\Lambda\rangle = 0, \forall|\alpha| \neq |\beta|$. Since all these terms are nonnegative, they obtain minimum at the maximal (minimal) weight state $\hat{\rho} = |\Lambda\rangle\langle\Lambda|$. Therefore, the whole expression (24) for the trace-norm of the covariance matrix obtains minimum at $\hat{\rho} = |\Lambda\rangle\langle\Lambda|$. □

4. Conclusions

The globally stable solutions of quantum dynamics modeled by the stochastic nonlinear Schrödinger equation (6) are the generalized coherent states, associated with the spectrum-generating algebra of the system. Stable solutions of the sNLSE in the case of Heiseberg–Weyl algebra have been obtained before (Diosi 1988b, Halliwell and Zoupas 1995, Schack *et al* 1995). Our result refers to a compact semisimple spectrum-generating algebra. The description by the stochastic nonlinear Schrödinger equation is equivalent to Lindblad semi-group modeling of the process of group-invariant weak measurement of the elements of the algebra. The Hamiltonian of the system is linear in the algebra elements, i.e., possesses dynamical symmetry. It is conjectured that breaking the symmetry by adding nonlinearity to the Hamiltonian results in the asymptotically stable localized solutions of the corresponding sNLSE (see (Khasin and Kosloff 2008) for some numerical evidence). The proof of stability is based on proving that the trace-norm of the covariance matrix, associated with the algebra, becomes minimal in a generalized coherent state.

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